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IOP PUBLISHING

J. Phys.: Condens. Matter **21** (2009) 445802 (7pp)

JOURNAL OF PHYSICS: CONDENSED MATTER

doi:10.1088/0953-8984/21/44/445802

Optical conductivity and electron–hole pair creation in graphene

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Abstract
The optical conductivity of graphene is studied in detail. It is shown how crossover from nonlinear to linear behaviour of the current density occurs when the frequency of the external electric field is increased. We also study electron–hole pair creation in bilayer graphene and show that its rate in a static electric field is determined by a power law as for single-layer graphene but with a different exponent. Subsequently, transport in carbon nanotubes is studied.

1. Introduction

One of the most amazing properties of graphene is that its zero-field conductivity does not disappear in the limit of vanishing carrier density but instead turns out to be of the order of the conductivity quantum e^2/h. However, it is the conductivity that is quantized in graphene, as opposed to the case for other transport phenomena, where the conductance is quantized.

The conductivity of graphene varies almost linearly with the carrier density [1, 2]. However, it exhibits a plateau near the neutrality point. There is much evidence that in early experiments [1, 2], Coulomb impurities on the substrate were the major impurities affecting the conductivity of graphene [3–5]. Numerical simulations [3] show that the minimum conductivity of graphene with point scatterers is close to $\frac{4}{\pi}\frac{e^2}{h}$ and it is approximately three times larger for Coulomb scatterers. The statistics of random voltage fluctuations generated by Coulomb impurities in the substrate was found in [4]. The transition to the linear regime when the gate voltage V_g (controlling the carrier density in graphene) increases can be viewed as a percolation transition in this random potential: paddles of electrons and holes generated in the random potential V increase with V_g and finally carriers of one type percolate and spread over the whole graphene sample. The mean square deviation $\sqrt{\overline{V^2}}$ determines the value of the gate voltage at which this happens. Electron–hole paddles were indeed observed in experiment [6]. More detailed study of the minimum conductivity, based on the picture described, was carried in [5]. All of this implies that the minimum conductivity of graphene is not universal as it might seem at first sight.

There are three theoretical values for the minimum conductivity of graphene:

$$\sigma_1^{\min} = \frac{4}{\pi}\frac{e^2}{h} \tag{1}$$

[7–15],

$$\sigma_2^{\min} = \frac{\pi}{2}\frac{e^2}{h} \tag{2}$$

[7, 13, 16–18] and

$$\sigma_3^{\min} = \pi\frac{e^2}{h} \tag{3}$$

[19]. It was shown that the minimum conductivity of graphene is sensitive to the order of taking certain limits (notably, $\omega \to 0$ and $\eta \to 0$, where ω is the frequency of the external electric field and η is the impurity scattering rate; see section 4), and all three values (1)–(3) were obtained in this way [20]. $\sigma_1^{\min}$ was obtained from the Kubo [7, 8, 10, 11, 14, 15] and the Landauer [9, 12, 15] formulae. $\sigma_{2,3}^{\min}$ were obtained from the Kubo formula only. In [18], the current was obtained by studying the dynamics of electron–hole pair creation after switching on an electric field. Since neither the Kubo formula nor the Landauer formula was used in that paper, all difficulties associated with regularization procedures fell away. $\sigma_2^{\min}$ was observed in experiments on light scattering by graphene [21, 22]. $\sigma_1^{\min}$ was also experimentally observed [23–26].

Experimental and theoretical studies concerning bilayer graphene are much less common than ones concerning single-layer graphene. Koshino and Ando [27] using the self-consistent Born approximation found that for dirty graphene $\sigma_{\min} = \frac{8}{\pi}\frac{e^2}{h}$, and in the opposite limit of clean graphene $\sigma_{\min} = \frac{24}{\pi}\frac{e^2}{h}$, which is six times larger than for single-layer

graphene. Katsnelson [28] used the Landauer formula and obtained the completely different result $\sigma_{\min} = 2e^2/h$. Cserti [13] found that $\sigma_{\min} = \frac{4J}{\pi}\frac{e^2}{h}$ for dirty and $\sigma_{\min} = \frac{\pi J}{2}\frac{e^2}{h}$ for clean graphene using the Kubo formula ($J = 1$ for single-layer graphene and $J = 2$ for bilayer graphene).

For a strong electric field E, the current in the system of two-dimensional Dirac fermions is determined by the creation of electron–hole pairs with the rate $\propto E^{\frac{3}{2}}$ (see section 5). This phenomenon is called the Schwinger mechanism [29–31]. This implies that the result of [18] is not applicable when the frequency of the external electric field $\omega \to 0$. Therefore, one has to study the method of [18] in more detail. After giving a modification of that method in section 2, we will present some of its applications in sections 3 and 4. Then we study the Schwinger mechanism in bilayer graphene in section 5 (without screening effects). We mention here that the consideration of the bilayer graphene in this paper is limited to the simplest case of unbiased graphene, when there is no gap between the parabolic bands. In section 6, we study transport in carbon nanotubes. In the last section 7, a summary of the results obtained is given and conclusions are drawn.

2. Dynamics of electron–hole pair creation in graphene

The following method was used to calculate the minimum conductivity of graphene in [18]. A uniform electric field with the vector potential $\mathbf{A} = (cEt, 0, 0)$ is switched on at the moment $t = 0$ in the system of particles described by the Dirac equation. Then the evolution of the wavefunction of the particles with initial values (6) is studied, treating the electric field as a perturbation. The current density is given in the linear approximation by (16). The authors of this paper claim that their result gives the true value for the dc conductivity of graphene. However, as will be seen from the subsequent discussion, this method is not applicable for finite graphene samples, finite field strengths and low frequencies $\omega \to 0$. We will modify their method and study various consequences. This will allow for better understanding of the method and obtaining some results that cannot be found with the original formulation.

To treat single-layer and bilayer graphene on an equal footing, let us do the following [13]. We rewrite the Hamiltonian in a unified form:

$$\hat{H}_J = g \begin{pmatrix} 0 & (\hat{q}_x - i\hat{q}_y)^J \\ (\hat{q}_x + i\hat{q}_y)^J & 0 \end{pmatrix} \quad (4)$$

where $J = 1$, $g = v_F$ corresponds to single-layer graphene ($v_F \approx 10^6$ m s^{-1} is the Fermi velocity) and $J = 2$, $g = 1/2m$ corresponds to bilayer graphene (where m is the effective mass). Equation (4) can be written in a more convenient way as

$$\hat{H}_J = \Omega(\mathbf{q})\hat{\sigma} \quad (5)$$

by introducing the vector $\Omega(\mathbf{q}) = gq^J(\cos J\varphi, \sin J\varphi)$, where $\mathbf{q} = (q_x, q_y) = q(\cos\varphi, \sin\varphi)$, and $\hat{\sigma} = (\hat{\sigma}_x, \hat{\sigma}_y)$ are the Pauli matrices.

Let $\psi_0(\mathbf{q})$ be the eigenfunctions of the Hamiltonian (5). Then, in the absence of external fields,

$$\psi_0(\mathbf{q}) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-iJ\varphi} \\ 1 \end{pmatrix} \quad (6)$$

and the dispersion of particles $\varepsilon(\mathbf{q})$ is given by

$$\varepsilon(\mathbf{q}) = gq^J. \quad (7)$$

At the moment $t = 0$, a uniform electric field directed along the x-axis with the frequency ω and the scalar potential $Ex\cos\omega t$ is switched on. Then the spinor $\psi(t, \mathbf{q})$ obeys the following equation:

$$i\frac{\partial}{\partial t}\psi(t, \mathbf{q}) = \Omega(\mathbf{q})\hat{\sigma}\psi(t, \mathbf{q}) + ieE\cos\omega t\frac{\partial}{\partial q_x}\psi(t, \mathbf{q}) \quad (8)$$

where we used the momentum representation $i\partial/\partial q_x$ of the position operator $\hat{x}$ (we assume for brevity that $\hbar = 1$).

Performing the Laplace transform of (8) we get

$$ip\bar{\psi}(p, \mathbf{q}) - i\psi_0(\mathbf{q}) = \Omega(\mathbf{q})\hat{\sigma}\bar{\psi}(p, \mathbf{q})$$
$$+ \frac{1}{2}ieE\frac{\partial}{\partial q_x}\left(\bar{\psi}(p + i\omega, \mathbf{q}) + \bar{\psi}(p - i\omega, \mathbf{q})\right) \quad (9)$$

where $\overline{\psi}(p, \mathbf{q}) = \int_0^\infty \psi(t, \mathbf{q})e^{-pt}\,dt$. We are going to treat the term proportional to E in (9) as a perturbation; then we will find the conditions when this procedure is justified. Then we obtain

$$\bar{\psi}(p, \mathbf{q}) = \frac{\psi_0(\mathbf{q})}{p + i\varepsilon(\mathbf{q})} + \frac{eE}{p + i\Omega(\mathbf{q})\hat{\sigma}}\frac{\partial}{\partial q_x}\frac{1}{2}\psi_0(\mathbf{q})$$
$$\times \left(\frac{1}{p + i\varepsilon(\mathbf{q}) + i\omega} + \frac{1}{p + i\varepsilon(\mathbf{q}) - i\omega}\right). \quad (10)$$

We represent the Laplace transform of the spinor $\bar{\psi}(p, \mathbf{q})$ in the form

$$\bar{\psi}(p, \mathbf{q}) = \frac{\psi_0(\mathbf{q})}{p + i\varepsilon(\mathbf{q})} + eE\overline{\xi}(p, \mathbf{q}). \quad (11)$$

It is convenient to rewrite $\overline{\xi}(p, \mathbf{q})$ in the form

$$\overline{\xi}(p, \mathbf{q}) = -i\frac{\partial\varepsilon(\mathbf{q})}{\partial q_x}\frac{\psi_0(\mathbf{q})}{p + i\varepsilon(\mathbf{q})}\frac{1}{2}\left(\frac{1}{(p + i\varepsilon(\mathbf{q}) + i\omega)^2}\right.$$
$$+ \frac{1}{(p + i\varepsilon(\mathbf{q}) - i\omega)^2}\right) + \frac{J}{2\sqrt{2}}\left(\frac{1}{p + i\varepsilon(\mathbf{q}) + i\omega}\right.$$
$$+ \frac{1}{p + i\varepsilon(\mathbf{q}) - i\omega}\right)\frac{iq_y}{q^2}\frac{1}{p^2 + \varepsilon(\mathbf{q})^2}$$
$$\times \left[p\begin{pmatrix} 1 \\ 0 \end{pmatrix} - i(\Omega_x(\mathbf{q}) + i\Omega_y(\mathbf{q}))\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]. \quad (12)$$

Taking the inverse Laplace transform of (12), we find

$$\xi(t, \mathbf{q}) = -i\frac{\partial\varepsilon(\mathbf{q})}{\partial q_x}\psi_0(\mathbf{q})$$
$$\times \left[\frac{1}{\omega^2}(1 - \cos\omega t) - \frac{1}{\omega}t\sin\omega t\right]\psi_0(\mathbf{q})e^{-i\varepsilon(\mathbf{q})t}$$
$$+ \frac{J}{4\sqrt{2}}e^{-iJ\varphi}\frac{iq_y}{q^2\varepsilon(\mathbf{q})}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\left[\frac{e^{i\varepsilon(\mathbf{q})t} - e^{-i\varepsilon(\mathbf{q})t - i\omega t}}{2i\varepsilon(\mathbf{q}) + i\omega}\right.$$
$$+ \frac{e^{i\varepsilon(\mathbf{q})t} - e^{-i\varepsilon(\mathbf{q})t + i\omega t}}{2i\varepsilon(\mathbf{q}) - i\omega} + 2e^{-i\varepsilon(\mathbf{q})t}\frac{\sin\omega t}{\omega}\right]$$

$$-\frac{J}{4\sqrt{2}}\frac{\mathrm{i}q_y}{q^2\varepsilon(\mathbf{q})}\begin{pmatrix}0\\1\end{pmatrix}\left[\frac{\mathrm{e}^{\mathrm{i}\varepsilon(\mathbf{q})t}-\mathrm{e}^{-\mathrm{i}\varepsilon(\mathbf{q})t-\mathrm{i}\omega t}}{2\mathrm{i}\varepsilon(\mathbf{q})+\mathrm{i}\omega}\right.$$

$$+\left.\frac{\mathrm{e}^{\mathrm{i}\varepsilon(\mathbf{q})t}-\mathrm{e}^{-\mathrm{i}\varepsilon(\mathbf{q})t+\mathrm{i}\omega t}}{2\mathrm{i}\varepsilon(\mathbf{q})-\mathrm{i}\omega}-2\mathrm{e}^{-\mathrm{i}\varepsilon(\mathbf{q})t}\frac{\sin\omega t}{\omega}\right]. \tag{13}$$

The current density is given by the following integral over the Brillouin zone:

$$j_x(t)=e\int_{\mathrm{BZ}}\frac{\mathrm{d}^2q}{(2\pi)^2}\psi(t,\mathbf{q})^\dagger\frac{\partial\boldsymbol{\Omega}(\mathbf{q})\hat{\sigma}}{\partial q_x}\psi(t,\mathbf{q}). \tag{14}$$

We are interested only in the term linear in electric field E in (14). Taking into account that initially there was no current in the system, i.e.

$$j_x(0)=e\int_{\mathrm{BZ}}\frac{\mathrm{d}^2q}{(2\pi)^2}\psi_0(\mathbf{q})^\dagger\frac{\partial\boldsymbol{\Omega}(\mathbf{q})\hat{\sigma}}{q_x}\psi_0(\mathbf{q})=0 \tag{15}$$

we find

$$j_x(t)=e^2E\int_{\mathrm{BZ}}\frac{\mathrm{d}^2q}{(2\pi)^2}\left(\psi(t,\mathbf{q})^\dagger\frac{\partial\boldsymbol{\Omega}(\mathbf{q})\hat{\sigma}}{\partial q_x}\xi(t,\mathbf{q})+\mathrm{h.c.}\right) \tag{16}$$

where h.c. denotes the Hermitian conjugate.

Substituting (13) into (16), proceeding to polar coordinates, and changing the variables $\varepsilon=gq^J$, we get

$$j_x(t)=\frac{e^2EJ}{2\pi h}\int_0^\infty\mathrm{d}\varepsilon\int_0^{2\pi}\sin^2\varphi\mathrm{d}\varphi$$

$$\times\left[\frac{\sin\left(\varepsilon-\frac{\omega}{2}\right)t\cos\left(\varepsilon+\frac{\omega}{2}\right)t}{2\varepsilon-\omega}\right.$$

$$+\left.\frac{\sin\left(\varepsilon+\frac{\omega}{2}\right)t\cos\left(\varepsilon-\frac{\omega}{2}\right)t}{2\varepsilon+\omega}\right]. \tag{17}$$

Noting that the second term in brackets in (17) can be obtained from the first by making the substitution $\varepsilon\rightarrow-\varepsilon$ and performing angle integration, we obtain

$$j_x(t)=\frac{e^2EJ}{2h}\int_{-\infty}^\infty\mathrm{d}\varepsilon\frac{\sin\left(\varepsilon-\frac{\omega}{2}\right)t\cos\left(\varepsilon+\frac{\omega}{2}\right)t}{2\varepsilon-\omega}. \tag{18}$$

Using the elementary formula $\cos(\varepsilon+\frac{1}{2}\omega)t=\cos(\varepsilon-\frac{1}{2}\omega)t\cos\omega t-\sin(\varepsilon-\frac{1}{2}\omega)t\sin\omega t$, we finally come to

$$j_x(t)=\frac{e^2EJ}{h}\int_{-\infty}^\infty\frac{\sin(2\varepsilon-\omega)t}{2\varepsilon-\omega}\cos\omega t\,\mathrm{d}\varepsilon$$

$$=\frac{e^2E}{h}\frac{\pi J}{2}\cos\omega t \tag{19}$$

(here we took into account the additional factor 4, due to spin and valley degeneracy).

Let us make an interesting observation concerning the factor $\sin(2\varepsilon-\omega)t/(2\varepsilon-\omega)$ in (19). When $t\rightarrow\infty$ this factor tends to $\pi\delta(\varepsilon-\omega/2)$. This means that in the limit $t\rightarrow\infty$ only the vicinity of $\varepsilon\approx\omega/2$ is substantial in the integral in (19). This has a simple explanation: in the case of normal incidence, when an electron in a valence band absorbs a photon, the electron momentum is unchanged because the wavevector of the incident wave is normal to the graphene surface (in general, the change in momentum of the electron is of the order of $\omega/c\ll\omega/v_\mathrm{F}$ and can be neglected). Therefore,

the electron in the valence band should have energy $-\omega/2$ to absorb a photon of frequency ω and transform to an electron in the conduction band with energy $\omega/2$ [22]. This fact will be useful in section 3 where we prove that the Fourier transform in (8) is justified for high frequencies and large sample sizes.

One can treat the term in (9) proportional to E as a perturbation if $eE\xi(p,\mathbf{q})$ is much less than $\psi_0(\mathbf{q})/(p+\mathrm{i}\varepsilon(\mathbf{q}))$ at momenta $gq^J\sim\omega$, which means $eE/\omega\ll(\omega/g)^{\frac{1}{J}}$. Introducing the electrical length $l_\mathrm{el}=(g/eE)^{\frac{1}{J+1}}$ ($l_\mathrm{el}=\sqrt{v_\mathrm{F}\hbar/eE}$ for single-layer graphene and $l_\mathrm{el}^\mathrm{b}=(meE)^{-\frac{1}{3}}$ for bilayer graphene) and group velocity $u=|\partial\varepsilon(\mathbf{q})/\partial q|\sim g(\omega/g)^{\frac{J-1}{J}}$, one can restate this condition in a more transparent form as $l_\mathrm{el}\gg u/\omega$. For example, in the case of single-layer graphene, it has a simple explanation: in an electrostatic field, pairs of electrons and holes are created with the rate $\propto E^{\frac{3}{2}}$, if the system size exceeds the electrical length l_el [29]. However, for an electric field with a high enough frequency $\omega\gg v_\mathrm{F}/l_\mathrm{el}$ (the group velocity for single-layer graphene is simply the Fermi velocity v_F), the sign of the electric field changes before the electrons have travelled a distance of the order of l_el. Therefore, we get a linear current in the system. Thus, one solves the question of the relation of the picture described to the Schwinger mechanism suggested in [18].

In order to account for the temperature dependence of the conductivity, one should introduce the factor $\tanh(\varepsilon/2T)$ in the integral in (18). This gives an extra factor $\tanh(\omega/4T)$ in (19). When $\omega\gg T$ one finds (19).

3. Finite graphene samples

In this section, we will study finite graphene samples with length L. For definiteness, we will consider the single-layer graphene. Initially the spinor $\psi(t,\mathbf{q})$ is equal to

$$\psi_0(\mathbf{q})=\frac{1}{\sqrt{2}}\begin{pmatrix}\frac{q_x-\mathrm{i}q_y}{\varepsilon(\mathbf{q})}\\1\end{pmatrix} \tag{20}$$

where $\varepsilon(\mathbf{q})=\sqrt{q_x^2+q_y^2}$ (we will drop the Fermi velocity v_F in some of the formulae; it can be easily restored in each case and this will not lead to confusion). At the moment $t=0$, a uniform electric field $E(t)=E\cos\omega t$ with the frequency ω and the scalar potential

$$U(x,t)=\begin{cases}-\dfrac{E(t)L}{2}&\text{for }-\infty<x<-\frac{L}{2}\\[2mm]E(t)x&\text{for }-\frac{L}{2}<x<\frac{L}{2}\\[2mm]\dfrac{E(t)L}{2}&\text{for }\frac{L}{2}<x<\infty\end{cases} \tag{21}$$

is switched on. The frequency ω determines the characteristic energy scale in the system. It will turn out that in the limit $\omega L\gg v_\mathrm{F}$ the frequency dependence of the conductivity disappears for large times $\omega t\gg1$. The Fourier transform of the Dirac equation for spinor $\psi(t,\mathbf{q})$ now takes the form

$$\mathrm{i}\frac{\partial}{\partial t}\psi(t,\mathbf{q})=q_x\hat{\sigma}_x\psi(t,\mathbf{q})+q_y\hat{\sigma}_y\psi(t,\mathbf{q})$$

$$-\frac{eE(t)L}{2}\int_{-\infty}^{-\frac{L}{2}}\mathrm{e}^{-\mathrm{i}q_xx}\psi(t,x,q_y)\,\mathrm{d}x$$

$$+ eE(t) \int_{-\frac{L}{2}}^{\frac{L}{2}} x \, \mathrm{e}^{-\mathrm{i}q_x x} \psi(t, x, q_y) \, \mathrm{d}x$$

$$+ \frac{eE(t)L}{2} \int_{\frac{L}{2}}^{\infty} \mathrm{e}^{-\mathrm{i}q_x x} \psi(t, x, q_y) \, \mathrm{d}x. \tag{22}$$

Using the representation

$$\psi(t, x, q_y) = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \mathrm{e}^{\mathrm{i}kx} \psi(t, k, q_y) \tag{23}$$

and integrating with respect to x using the formulae

$$\int_{-\infty}^{-\frac{L}{2}} \mathrm{e}^{\mathrm{i}(k-q)x} \mathrm{d}x = \frac{\mathrm{i}\mathrm{e}^{-\frac{\mathrm{i}L(k-q)}{2}}}{q-k+\mathrm{i}\delta} = \frac{\mathrm{i}\mathrm{e}^{-\frac{\mathrm{i}L(k-q)}{2}}}{q-k} + \pi\delta(k-q) \tag{24}$$

$$\int_{\frac{L}{2}}^{\infty} \mathrm{e}^{\mathrm{i}(k-q)x} \mathrm{d}x = \frac{\mathrm{i}\mathrm{e}^{\frac{\mathrm{i}L(k-q)}{2}}}{k-q+\mathrm{i}\delta} = \frac{\mathrm{i}\mathrm{e}^{\frac{\mathrm{i}L(k-q)}{2}}}{k-q} + \pi\delta(k-q) \tag{25}$$

we obtain that the sum of three integrals in (22) transforms to the form

$$\mathrm{i}eE(t) \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \psi(t, k, q_y)$$

$$\times \left(L \frac{\cos \frac{(k-q_x)L}{2}}{k-q_x} + 2 \frac{\partial}{\partial q_x} \frac{\sin \frac{(k-q_x)L}{2}}{k-q_x} \right). \tag{26}$$

In the limit $\omega L \gg v_F$, for $k, q \sim \omega$ (as we have seen in section 2), we can substitute $\sin \frac{(k-q)L}{2}/(k-q)$ by $\pi\delta(k-q)$. Then the second term in (26) gives exactly $\mathrm{i}eE\partial/\partial q_x$ and the first is of the order of $v_F/\omega L$ and can be neglected. Finally, (22) reduces to (8).

We can conclude that for large sample sizes and high frequencies $\omega L \gg v_F$ the minimum conductivity of graphene is $\sigma_2^{\min} = \frac{\pi}{2} \frac{e^2}{h}$. This can be qualitatively explained from the point of view of Ziegler's work [20]. If one identifies the impurity scattering rate η with the frequency v_F/L of scattering by the boundaries of the sample, then the limit $\omega \gg \eta$, in which the Kubo formula gives the conductivity $\sigma_2^{\min}$, is equivalent to $\omega L \gg v_F$. For the particle to feel the frequency dependence of the external electric field, the frequency ω must be larger than the impurity scattering rate η or the frequency of scattering by the boundaries v_F/L. Therefore, one can conclude that in the inverse limit $\omega L \ll v_F$ the minimum conductivity should be $\sigma_1^{\min} = \frac{4}{\pi} \frac{e^2}{h}$. However, we will see that a special configuration is needed for this to happen.

Let us check whether the validity conditions of the theory are satisfied in experiments on light scattering by graphene [21, 22]. The width of the light beam is $L = 30 \, \mu\mathrm{m}$. The light frequency $\omega \approx 500$ THz. The strength of the electric field in the light wave $E = 1000 \, \mathrm{V \, m^{-1}}$. This gives for the electrical length $l_{\mathrm{el}} \approx 1 \, \mu\mathrm{m}$ and for another length scale in the problem $l_\omega = v_F/\omega \approx 1$ nm. We see that the conditions $l_{\mathrm{el}} \gg v_F/\omega$, $L \gg v_F/\omega$ are satisfied. Let us check the analogous conditions for bilayer graphene. The effective mass is approximately $m \approx 0.044 m_e$, where m_e is the electronic mass [32]. This gives for the electrical length $l_{\mathrm{el}}^b \approx 100$ nm and for the length scale $l_\omega = \sqrt{\hbar/m\omega} \approx 1$ nm. The conditions $l_\omega \ll L, l_\omega \ll l_{\mathrm{el}}^b$ are also satisfied in this case.

This modified method allows not only for a more rigorous derivation of the minimum conductivity of graphene but also for one to go further and investigate an interesting case (see section 4).

4. Crossover from $\sigma_1^{\min}$ to $\sigma_2^{\min}$

The following configuration is used to measure the conductivity of graphene in actual experiments. It was theoretically studied in [9, 12]. A graphene ribbon of length L and width $W \gg L$ is connected to two doped electrodes with potential $V_0 < 0$ and Fermi energy $E_F = v_F k_F = -V_0$ at its boundaries $x = \pm L/2$. The ribbon is kept at the neutrality point at which the carrier density equals zero. If $k_F L \gg 1$ then the transmission probabilities computed in [9, 12] are given by $T_n = 1/\cosh^2(k_n L)$, where $k_n = 2\pi n/W$, $n = 0, \pm 1, \pm 2, \ldots$. Substituting them in the Landauer formula

$$G = \frac{e^2}{h} \sum_n T_n \tag{27}$$

one obtains for the conductance $G = \frac{4e^2}{\pi h} \frac{W}{L}$ and consequently for the conductivity $\sigma = \frac{4}{\pi} \frac{e^2}{h}$.

The Landauer formalism cannot be applied for high frequencies $\omega L \gg v_F$ [33]. However, this is exactly the case considered in section 3. We can try to apply the same considerations to the present case. The Fourier transform of the Dirac equation for electrons moving in a potential

$$V(x) = \begin{cases} V_0 & \text{for } -\infty < x < -L/2 \text{ or } L/2 < x < \infty \\ 0 & \text{for } -L/2 < x < L/2 \end{cases} \tag{28}$$

is

$$\varepsilon(\mathbf{q})\psi_0(\mathbf{q}) = q_x \hat{\sigma}_x \psi_0(\mathbf{q}) + q_y \hat{\sigma}_y \psi_0(\mathbf{q}) + V_0 \psi_0(\mathbf{q})$$

$$- 2V_0 \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \psi_0(q_x - k, q_y) \frac{\sin \frac{kL}{2}}{k}. \tag{29}$$

One can, at least in principle, obtain the dispersion relation $\varepsilon(\mathbf{q})$ from this linear integral equation. However, we are interested only in the solution of this equation for high frequencies. If $qL \gg 1$ and $\psi_0(\mathbf{q})$ is a sufficiently smooth function, then $\sin(kL/2)/k$ in (29) can be substituted by $\pi\delta(k)$ with accuracy of the order of $1/qL$. For $V_0/qL \ll q$, one obtains that the potential $V(x)$ is smoothed and, therefore, can be thrown away. This heuristic derivation was based on the assumption that the spinor $\psi(\mathbf{q})$ is a smooth function of the momentum $\mathbf{q}$, and ultimately we obtained that it is indeed smooth and is given by ψ_0 from section 2, while the dispersion relation is given by $\varepsilon(q, k) = \sqrt{q^2 + k^2}$. To calculate the conductivity at the frequency ω, one needs the solution to (29) for momenta $qv_F \sim \omega$. Now we can apply the results of two previous sections and obtain the following result.

When the following conditions hold:

$$\omega/v_F, k_F \gg L \quad k_F v_F^2 \ll \omega^2 L \tag{30}$$

the conductivity of a graphene ribbon equals $\sigma_2^{\min} = \frac{\pi}{2} \frac{e^2}{h}$, i.e. there is a crossover from $\sigma_1^{\min}$ to $\sigma_2^{\min}$ when the frequency of the external field increases.

Ziegler showed [20] that an analogous crossover exists in dirty graphene: the conductivity of graphene is $\sigma_1^{\min}$ for $\omega \ll \eta$ and $\sigma_2^{\min}$ for $\omega \gg \eta$. Here we have shown how it occurs

in clean graphene samples when the parameters of the system vary.

It seems that this result is of interest from the theoretical point of view only because the finite capacitance of the system, which affects the ac conductivity, could complicate the problem of detection of this crossover in experiments. Besides, $\sigma_1^{\min}$ is numerically close to $\sigma_2^{\min}$.

5. p–n junctions in bilayer graphene

The current through a wide graphene p–n junction is given by

$$I_{\text{p–n}} = \frac{e^2 L}{\pi^2} \sqrt{\frac{eE^3}{v_{\text{F}}}}. \tag{31}$$

Equation (31) can be obtained by finding coefficients of transmission of electrons through the potential barrier formed between the electrodes [34]. It can also be obtained by calculating the rate of electron–hole pair creation in graphene in a uniform electric field [29]. The formula (31) is valid only for wide and long p–n junctions for which $W, L \gg l_{\text{el}}$.

Here we will study p–n junctions in bilayer graphene. First we note that the transmission probabilities found in [34] are equal to (see also [35])

$$T(k) = e^{-\frac{\pi k^2}{eE}} = e^{-\pi(kl_{\text{el}})^2}. \tag{32}$$

In analogy with this formula, remembering that the electrical length for bilayer graphene is $l_{\text{el}}^{\text{b}} = (meE)^{-\frac{1}{3}}$, one can assume that for a bilayer graphene p–n junction the transmission probabilities are given by

$$T_{\text{b}}(k) = f(kl_{\text{el}}^{\text{b}}) \tag{33}$$

where f is an unknown function. The conductance of the junction can be found using the Landauer formula

$$G = 4\frac{e^2}{h} \sum_{-k_{\text{F}} < k < k_{\text{F}}} T(k). \tag{34}$$

For a wide p–n junction $W \gg l_{\text{el}}^{\text{b}}$, one obtains

$$G = \frac{2e^2 W}{\pi h} \int_{-k_{\text{F}}}^{k_{\text{F}}} T(k) \, dk. \tag{35}$$

It is easily seen that the conductance of the junction is proportional to $E^{\frac{1}{3}}$: $G \propto E^{\frac{1}{3}}$. Now we can estimate the rate of pair creation Γ_{b} in bilayer graphene: $I = GEL = \Gamma_{\text{b}} WL$, from which it follows that $\Gamma_{\text{b}} \sim E^{\frac{4}{3}}$. From a slightly different point of view, p–n junctions in bilayer graphene are considered in [36].

Transmission probabilities $T_{\text{b}}(k)$ can be calculated in a quasiclassical approximation, in analogy with the single-layer graphene case [36]. First we proceed to a representation where the coordinate x plays the role of time, and the initial problem is represented in the form of an evolution with the non-Hermitian Hamiltonian $i\partial/\partial x = \hat{H}(x)$. The x-dependent eigenvalues of this Hamiltonian are given by $\kappa(x) = \pm\sqrt{k^2 - 2m|\varepsilon - eEx|}$. This quantity is imaginary

in the classically forbidden region $x_1 < x < x_2$, $x_{1,2} = \pm k^2/(2meE)$ (we set $\varepsilon = 0$ for convenience). Now we apply the adiabatic approximation. Transmission probabilities are given with exponential accuracy by the formula e^{-S}, where

$$S = 2 \int_{x_1}^{x_2} \text{Im} \, \kappa(x) \, dx = \frac{4}{3}(l_{\text{el}}^{\text{b}} k)^{\frac{1}{3}}. \tag{36}$$

We see that they have the form (33), the integral in (35) being convergent for large momenta k.

6. Carbon nanotubes

The electron–hole pair creation rate in a static electric field E for particles described by the Dirac equation is given by (per conduction channel)

$$\Gamma_1 = \frac{eE}{2\pi} \tag{37}$$

in one dimension, $D = 1$, and by

$$\Gamma_2 = \frac{(eE)^{\frac{3}{2}}}{4\pi^2} \tag{38}$$

in two dimensions, $D = 2$ [30]. For a long $L \gg l_{\text{el}}$ ribbon with arbitrary width W,

$$\Gamma_{\text{ribbon}} = \frac{eE}{2\pi W} \sum_n e^{-\frac{\pi k_n^2}{eE}} \tag{39}$$

where k_n are the quantized values of the transverse momenta. Now we will show how crossover occurs from (38) to (37) when the ribbon is narrowed. If $W \gg l_{\text{el}}$ then the sum in (39) can be replaced by an integral

$$\frac{eE}{2\pi} \int \frac{dk}{2\pi} e^{-\frac{\pi k^2}{eE}} = \frac{(eE)^{\frac{3}{2}}}{4\pi^2}. \tag{40}$$

However, if there is a zero mode and the contributions of the other modes are exponentially suppressed in weak fields due to a gap of the order of v_{F}/W, i.e. $W \ll l_{\text{el}}$, then only one term with $n = 0$ remains in (39):

$$\Gamma_2 = \frac{eE}{2\pi W}. \tag{41}$$

Multiplying by W, one obtains Γ_1.

The Kubo formula gives the result e^2/h for the conductance of ballistic wires [37] (per conduction channel). However, the Kubo formula is obtained from linear response theory and therefore valid only in zero electric field. In the opposite case of strong electric fields, one can deduce from (37) that the conductance is the same, e^2/h (to obtain the conductance from (37) one should multiply (37) by the length of the wire L and divide the result by the voltage drop $V = EL$).

What happens for arbitrary values of the electric field $l_{\text{el}} \sim L$? From the Luttinger liquid theory, it follows that the conductance does not depend on the magnitude of the electric field. Let us give here the proof of this fact taken from [37]

(with slight modifications). The action for the Luttinger liquid interacting with an electromagnetic field has the form

$$S = \frac{1}{2} \int dx \, dt \left[v_F^2(\partial_x \varphi)^2 - (\partial_t \varphi)^2 \right.$$
$$\left. + \frac{e}{\sqrt{\pi}}(A_0 \partial_x \varphi - A_x \partial_t \varphi) \right] \tag{42}$$

where A_μ is the vector potential of the electromagnetic field, and $\rho = \partial_x \varphi / \sqrt{\pi}$, $j = -\partial_t \varphi / \sqrt{\pi}$ are, respectively, charge and current densities. Varying φ, one obtains the equations of motion

$$\partial_t^2 \varphi - \partial_x^2 \varphi = \frac{e}{\sqrt{\pi}} E(x) \tag{43}$$

where $E(x) = -\partial_x A_0 + \partial_t A_x$ is the strength of the electric field. We will assume that it is not zero only in the region $0 < x < L$ and is switched on at $t = 0$. Then performing the Laplace transform of (43) gives

$$p^2 \overline{\varphi} - \partial_x^2 \overline{\varphi} = \frac{e}{\sqrt{\pi}} E(x) \frac{1}{p}. \tag{44}$$

The general solution to this equation is

$$\varphi(p, x) = A(p)e^{px} + B(p)e^{-px}$$
$$- \frac{e}{\sqrt{\pi} p^2} \int_{-\infty}^{x} E(y) \sinh(x - y) p \, dy \tag{45}$$

where $A(p)$ and $B(p)$ are arbitrary constants. The solution to this equation, limited at $x \to \pm\infty$, is

$$\varphi(p, x) = \frac{e}{2\sqrt{\pi} p^2} \int_0^L E(y)e^{-py} \, dy$$
$$- \frac{e}{\sqrt{\pi} p^2} \int_0^x E(y) \sinh(x - y) p \, dy. \tag{46}$$

The Laplace transform of the current density $j(t, 0) = -\partial_t \varphi(t, 0)/\sqrt{\pi}$, whose initial value is set to zero, is given by

$$j(p, 0) = \frac{p}{\sqrt{\pi}} \varphi(p, 0) = \frac{e}{2\sqrt{\pi} p} \int_0^L E(y)e^{-py} \, dy. \tag{47}$$

Using the Laplace transform of the step function $\theta(t - y) = e^{-py}/p$, we obtain $j = e^2 V/2\pi$ for $t > L/v_F$, where $V = \int_0^L E(y) \, dy$.

This result could be obtained from the Landauer formula, since the probability of transmission of Dirac quasiparticles through a barrier is unity for normal incidence (the Klein paradox).

This derivation for the Luttinger liquid shows that there is in principle no difference for the cases of strong ($l_{el} \ll L$) and weak ($l_{el} \gg L$) electric fields.

7. Conclusions

In summary, we have studied the optical conductivity of single-layer and bilayer graphene. To do this a modification of the method proposed in [18] was used. This modification allowed us to investigate how the ac electric field affects the dynamics of the electron–hole pair creation in graphene: there is a crossover from a nonlinear to a linear regime when the frequency of the external electric field is increased. It also

allowed to consider some finite size effects and to propose a mechanism of transition from $\sigma_1^{\min}$ to $\sigma_2^{\min}$ in clean graphene. In order to achieve this transition a special configuration needs to be considered (no such configuration is needed for dirty graphene [20]). In section 5 p–n junctions in bilayer graphene are studied and it is shown that the rate of electron–hole pair creation in a static electric field is proportional to $E^{\frac{4}{3}}$, whereas for single-layer graphene it is proportional to $E^{\frac{3}{2}}$. It is not known how screening effects and generating a gap in bilayer graphene (by applying an electric field perpendicular to the graphene surface) modify this result. Then a study of transport in carbon nanotubes is presented. Since the conductivities for strong ($l_{el} \ll L$) and weak ($l_{el} \gg L$) electric fields are the same (e^2/h per conduction channel), a question arises: what is the conductivity for intermediate values of the electric field? Using the Luttinger liquid theory one can show that the conductivity does not depend on the electric field strength. It seems, therefore, that there are no nonlinear effects in carbon nanotubes (at least in the approximation where the electrons are treated as non-interacting and where the linear dispersion of quasiparticles is justified).

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